Symmetry TFT and 4D supersymmetric field theory

Qiang Jia

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Korea Institute for Advanced Study
with Zhihao Duan and Sungjay Lee

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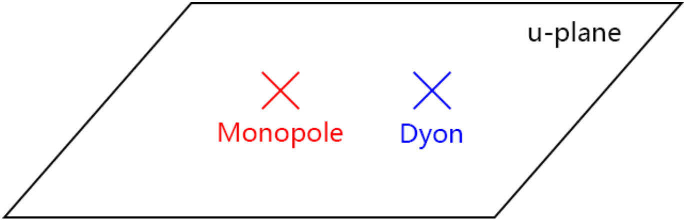
4D supersymmetric gauge theory

- In the past decades, a lot of progress has been made in the study of supersymmetric theories.
- People have invented a plethora of protected quantities that capture the dynamics of the system
- Supersymmetric indices
  - Witten index [Witten,1982]
  - Superconformal index
  - ...
- Topological invariants [Witten,1988]
  - Donaldson-Witten invariants
  - Vafa-Witten invariants [Vafa,Witten,1994] [Manschot,Moore,2021]
- Those quantities depend on the global structure of the gauge group.
- $SU(2)$ vs $SO(3)$, $SU(4)$ vs $SO(6)$ for example.

Witten index on T^4 [Witten,2002] [Tachikawa,2014]

- Consider 4D $N = 1$ pure SU(2) theory, there are two supersymmetric vacua [Seiberg,Witten,1994]



- The Witten index on T^4 (large radius) is

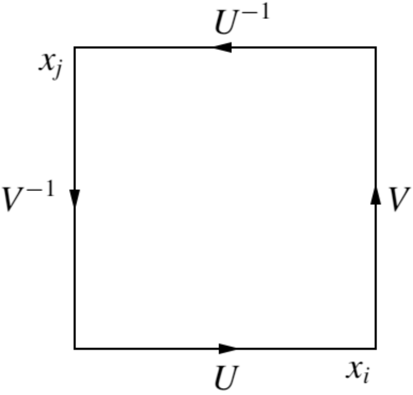
$$\mathcal{I}_{\mathrm{SU}(2)} = \mathrm{Tr}(-1)^F e^{-\beta H} = 1 + 1 = 2$$

Witten index on T^4

- Consider the gauge group to be $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ instead.
- On R^4, the local dynamics are the same, we still have two vacua.
- This is not true for T^4, one actually gets

$$\mathcal{I}_{\mathrm{SO}(3)} = 2 + 7 = 9 = 8 + 1$$

- From Hamiltonian point of view, the extra seven states are contributed by the non-trivial flat SO(3) bundle characterized by the discrete 't Hooft flux.



- Consider a T^2 parametrized by $x_i, x_j (i, j = 1, 2, 3)$ and denote the holonomy along x_i, x_j as $U, V \in SU(2)$ and consider

$$V^{-1}U^{-1}VU = (-1)^{\omega_{ij}}$$

- If the gauge group is SU(2), for a flat configuration one must have $\omega_{ij} = 0$.
- However, if the gauge group is SO(3), $\omega_{ij} = 1$ is also acceptable, since $-1 \in SU(2)$ projects to $1 \in SO(3)$.

- $\omega_{ij} \in \mathbb{Z}_2$ is the obstruction of lifting SO(3)-bundle to SU(2)-bundle.
- On spatial T^3, there are totally $2^3 - 1 = 7$ non-trivial flat SO(3) configuration characterized by $\omega_{12}, \omega_{23}, \omega_{13}$. Each contributes one vacuum.
- ω_{ij} are also known as discrete 't Hooft flux. Originally, it is realized by imposing a twist boundary condition for SU(2) gauge field ['t Hooft, 1980]

$$A_\nu(x_\mu = a_\mu) = \Omega_\mu \left(A_\nu(x_\mu = 0) - i\frac{\partial}{\partial x_\nu} \right) \Omega_\mu^{-1}$$

  where $A_\nu(x_\mu = a_\mu)$ and $A_\nu(x_\mu = 0)$ are glued via a gauge transformation $\Omega_\mu(x \neq x_\mu)$
- ω_{ij} are encoded as

$$(-1)^{\omega_{ij}} = \Omega_i^{-1}(x_j = 0)\Omega_j^{-1}(x_i = a_i)\Omega_i(x_j = a_j)\Omega_j(x_i = 0)$$

  which are gauge invariant quantities.

Discrete 't Hooft flux as 2-form background

- Recall that, if we have a complex scalar $\phi(\theta)$ living along a circle with a twist boundary condition

$$\phi(\theta + 2\pi) = e^{i\omega}\phi(\theta)$$

  If we consider a singular gauge transformation

$$\phi(\theta) \to e^{-\frac{\omega\theta}{2\pi}}\phi, \quad A_\theta \to A_\theta + \frac{\omega}{2\pi}$$

  then the scalar is periodic at the expense of introducing a $U(1)$ holonomies.

- Similarly, one can perform a singular gauge transformation to eliminate the twist boundary condition at the expense of introducing a 2-form background

$$b = \frac{1}{2}\sum_{i,j}\omega_{ij}dx_i \wedge dx_j$$

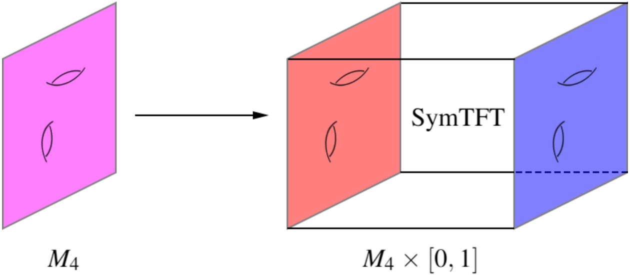
- Discrete 't Hooft flux $\leftrightarrow$ 2-form background

- Given a theory with gauge group G, we may consider its maximally covering group $\hat{G}$ with 2-form background $b \in H^2(M_4, Z(\hat{G}))$ and denote the corresponding supersymmetric quantity as $\mathcal{I}_{\rm SUSY}[b]$
- $\mathcal{I}_{\rm SUSY}[b]$ carries all information such that, for any G sharing the same Lie algebra, one has

$$\mathcal{I}_{G,\varphi} = \frac{1}{\mathcal{N}} \sum_{b \in H^2(M_4, Z(\hat{G}/G))} e^{i\varphi(b)} \mathcal{I}_{\rm SUSY}[b]$$

  with $\mathcal{N}$ certain normalization factor, $\varphi(b)$ discrete torsion.
- In other words, gauging the 1-form symmetry changes the global structure of the gauge group [Gaiotto,Kapustin,Seiberg,Willett,2015]
- From this point of view, the problem is best formulated in terms of Symmetry Topological Field Theory (TFT).

Symmetry TFT [Lakshya,Sakura,2023 (A review)] [Witten,1998]



- In the present case, the SymTFT is 5D BF theory

$$S_{BF} = \frac{N}{2\pi} \int \widetilde{B} \wedge dB$$

- The 4D quantities can be expanded as

$$\mathcal{I}_{G,\varphi} = {}_{\text{top}}\langle G, \varphi | e^{iHt} | \chi_{\text{SUSY}} \rangle$$

- $|\chi_{\text{SUSY}}\rangle$ is "dynamics boundary state".
- $|G, \varphi\rangle_{\text{top}}$ is "topological boundary state".
- Gauging 1-form symmetry is amount to changing topological boundary state

Changing global structure of gauge group

↕

Gauging 1-form symmetry/summing over 2-form background

↕

Switching topological boundary state in SymTFT

- The motivation of this work is to study 4D supersymmetric invariants using SymTFT, focusing on the global structures of the gauge group.
- We consider three concrete examples
  - Witten index on T^4 (Spin)
  - Superconformal index on $L(r,1) \times S^1$ (Torsion) [Razamat,Willett,2013]
  - Vafa-Witten invariants on $\mathbb{CP}_2$ (Non-Spin)
- They are formulated on spin, non-spin and torsional manifold separately
- Through those examples, we will work out the details of topological/dynamical boundary state on various manifolds

5D BF theory as SymTFT

$$S_{BF} = \frac{N}{2\pi} \int_{M_4 \times [0,1]} \widetilde{B} \wedge dB$$

- Here B and $\widetilde{B}$ are two-form gauge fields.
- For any closed 2-cycle $\Gamma \in H_2(M_4)$, one can construct two gauge invariant surface operators

  $$U[\Gamma] = \exp\left[i \oint_\Gamma B\right], \quad \widetilde{U}[\Gamma] = \exp\left[i \oint_\Gamma \widetilde{B}\right]$$

  and they satisfy the quantum algebra

  $$U[\Gamma]\widetilde{U}[\Gamma'] = \omega^{-K(\Gamma,\Gamma')}\widetilde{U}[\Gamma']\, U[\Gamma]$$

  in the Hamiltonian picture. $K[\Gamma, \Gamma']$ is the intersection number and ω is N-root of unity.
- Moreover, they satisfy

  $$U^N[\Gamma] = \widetilde{U}^N[\Gamma] = \mathbf{1} \,,$$

Boundary states

- Topological boundary states are 1-1 corresponds to maximally commuting set of operators
- Among them, there are two canonical boundary state
  - Dirichlet boundary state $|b\rangle$ (Diagonalizing U)

$$U[\Gamma]|b\rangle = \omega^{\int \gamma\wedge b}|b\rangle, \quad \widetilde{U}[\Gamma]|b\rangle = |b-\gamma\rangle$$

  - Neumann boundary state $|\tilde{b}\rangle$ (Diagonalizing $\widetilde{U}$)

$$\widetilde{U}[\Gamma]|\tilde{b}\rangle = \omega^{\int \gamma\wedge \tilde{b}}|\tilde{b}\rangle, \quad U[\Gamma]|b\rangle = |b+\gamma\rangle$$

  - They are related by

$$|\tilde{b}\rangle = \frac{1}{\sqrt{N^{h_2}}}\sum_b \omega^{\int \tilde{b}\wedge b}|b\rangle$$

- Identify b as the 2-form background of the 4D theory, the dynamical boundary state is constructed as

$$|\chi_{\text{SUSY}}\rangle = \sum_b \mathcal{I}_{\text{SUSY}}[b]|b\rangle$$

$SL(2,\mathbb{Z}_N)$ and Pontryagin square

- The 5D BF theory is invariant under an $SL(2,\mathbb{Z}_N)$ transformation generated by S and T

$$
\begin{aligned}
S: &\quad B \to \widetilde{B}, \quad \widetilde{B} \to -B, \\
T: &\quad B \to B, \quad \widetilde{B} \to \widetilde{B} + B.
\end{aligned}
$$

  S-transformation switch $U/\widetilde{U}$

$$
V_S U[\Gamma] V_S^{\dagger} = \widetilde{U}[\Gamma], \quad V_S \widetilde{U}[\Gamma] V_S^{\dagger} = U[-\Gamma]
$$

  and T-transformation generates

$$
V_T U[\Gamma] V_T^{\dagger} = U[\Gamma], \quad V_T \widetilde{U}[\Gamma] V_T^{\dagger} = S_{(1,1)}[\Gamma]
$$

- Here the generic surface operator is

$$
S_{(e,m)}[\Gamma] = \exp\left[i \oint_{\Gamma} eB + m\widetilde{B} \right]
$$

- In particular, one has

$$S_{(1,1)}[\Gamma] = \exp\left[i \oint_\Gamma B + \widetilde{B}\right] = \omega^{\frac{1}{2}\int \mathfrak{P}(\gamma)} U[\Gamma]\widetilde{U}[\Gamma]$$

where $\mathfrak{P}(\gamma)$ is the Pontryagin square maps $H^2(M_4, \mathbb{Z}_N)$ to $H^4(M_4, \mathbb{Z}_{2N})$

$$\mathfrak{P}(\gamma) = \begin{cases} \gamma \cup \gamma \quad (N \text{ is odd}) \\ \gamma \cup \gamma + \gamma \cup_1 \delta\gamma \quad (N \text{ is even}) \end{cases}$$

- One can work out

$$\begin{cases} V_S|b\rangle = \frac{1}{\sqrt{N^{h_2}}} \sum_{b'} \omega^{K(b,b')}|b'\rangle = |\tilde{b} = b\rangle \\ V_T|b\rangle = \omega^{-\frac{1}{2}\int \mathfrak{P}(b)}|b\rangle \end{cases}$$

V_S switch D/N boundary state, V_T stack an SPT phase.

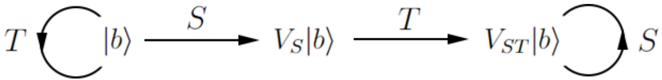


Figure: Topological boundary states for $N = 2$

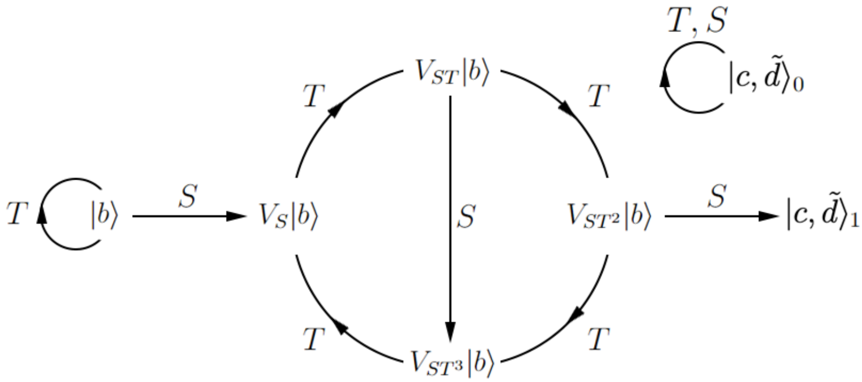


Figure: Topological boundary states for $N = 4$

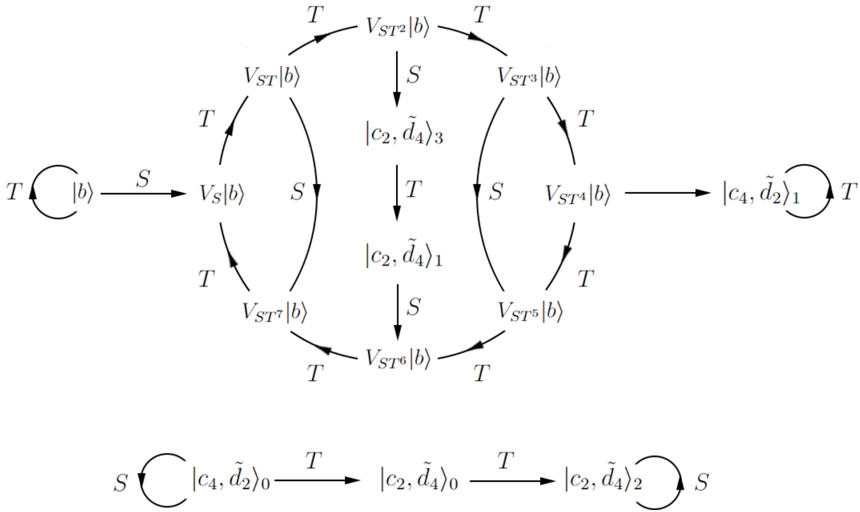


Figure: Topological boundary states for $N = 8$

Witten index on T^4

- On T^4, the 2-form b-field can be decomposed as

$$b = \sum_i t_i dx^0 \wedge dx^i + \frac{1}{2} \sum_{i,j,k} s_i \epsilon_{ijk} dx^j \wedge dx^k$$

  and we can denote $|b\rangle = |(t_1, t_2, t_3), (s_1, s_2, s_3)\rangle \equiv |(t, s)\rangle$
- S-transformation switch $|(t, s)\rangle$ and $|(\tilde{t}, \tilde{s})\rangle$

$$|(\tilde{t}, \tilde{s})\rangle = \frac{1}{N^3} \sum_{t,s} \omega^{\tilde{t}\cdot s + \tilde{s}\cdot t} |(t, s)\rangle$$

- T-transformation stack a phase

$$V_T |(t, s)\rangle = \omega^{-t\cdot s} |(t, s)\rangle$$

- The dynamics boundary states are constructed as following [Witten,2002]

$\hat{G}$	Center	Dynamical boundary state $\lvert\chi_{\text{SUSY}}\rangle$
$SU(n)$	$\mathbb{Z}_n$	$(-1)^{n-1} n \sum_{t,s} \delta_{t\cdot s,0} \lvert(t,s)\rangle$
$Sp(n)$	$\mathbb{Z}_2$	$(-1)^{n}(n+1) \sum_{t,s} \delta_{nt\cdot s,0} \lvert(t,s)\rangle$
$Spin(2n+1)$	$\mathbb{Z}_2$	$(-1)^{n}(2n-1) \sum_{t,s} \lvert(t,s)\rangle$
$Spin(4n+2)$	$\mathbb{Z}_4$	$-4n \sum_{t,s} \delta_{t\cdot s,0} \lvert(t,s)\rangle$
$Spin(8n+4)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(8n+2) \sum_{t,s;t',s'} \delta_{t\cdot s+t'\cdot s',0} \lvert(t,s);(t',s')\rangle$
$Spin(8n)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(8n-2) \sum_{t,s;t',s'} \delta_{t\cdot s'+t'\cdot s,0} \lvert(t,s);(t',s')\rangle$
E_6	$\mathbb{Z}_3$	$12 \sum_{t,s} \delta_{2t\cdot s,0} \lvert(t,s)\rangle$
E_7	$\mathbb{Z}_2$	$-18 \sum_{t,s} \delta_{t\cdot s,0} \lvert(t,s)\rangle$

- For example, for $SU(N)$ theory, the dynamical boundary state is

$$|\chi_{\mathrm{SUSY}}\rangle = (-1)^{N-1} N \sum_{t,s} \delta_{t\cdot s,0} |(t,s)\rangle$$

- The Witten index of $SU(N)$ is

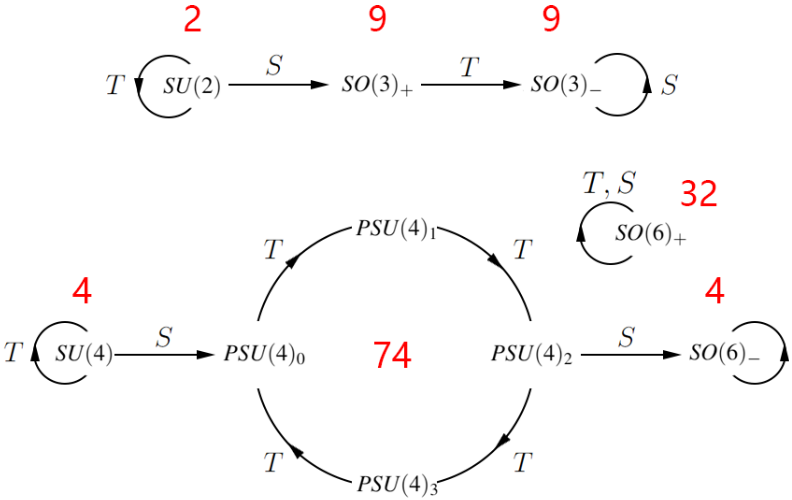
$$Z[t=0, s=0] \equiv \mathrm{Tr}(-1)^F = \langle (0,0)|\chi_{\mathrm{SUSY}}\rangle = (-1)^{N-1} N$$

- The Witten index of $SU(N)/\mathbb{Z}_N$ is

$$\langle (\tilde{0},\tilde{0})|\chi_{\mathrm{SUSY}}\rangle = (-1)^{N-1} \sum_{k=0}^{N-1} (\mathrm{gcd}(N,k))^3$$

- For $N=2$, one has

$$\langle (\tilde{0},\tilde{0})|\chi_{\mathrm{SUSY}}\rangle = -1 - 8 = -9$$



2
9
9
T
SU(2)
S
SO(3)+
T
SO(3)−
S
T, S
32
SO(6)+
PSU(4)1
T
T
4
4
T
SU(4)
S
PSU(4)0
74
PSU(4)2
S
SO(6)−
T
T
PSU(4)3

Superconformal index on $L(r, 1) \times S^1$

- Let's then consider the 4D $\mathcal{N} = 1$ superconformal index on $L(r, 1) \times S^1$

$$\mathcal{I} = \mathrm{Tr}\left[(-1)^F q^{\hat{D}-\frac{1}{2}\hat{R}} x^{2\hat{J}_R^3+\hat{R}} y^{2\hat{J}_L^3} e^{im\beta}\right],$$

- Using localization technique, the index can be reduced to an integral along the flat configuration, characterized by the holonomies.
- The holonomy along S^1 is denoted as U
- $L(r, 1) = S^3/\mathbb{Z}_r$ has a torsion 1-cycle C_τ such that $rC_\tau = 0$. We denote the holonomies along C_τ as V

- If we turn off the ’t Hooft flux, then U and V commute and both lie in the Cartan torus.
- However, since C_τ is torsion, one should have $V^r = 1$ and elements of V are discrete and are labelled by

$$\mathbf{m} = (m_1, m_2, \cdots, m_{\mathrm{rank}(G)})$$

- Then the index in the trivial sector is

$$\mathcal{I} = \sum_{\mathbf{m}} \frac{1}{|W(\mathbf{m})|} \oint \prod_{l=1}^{\mathrm{rank}(\hat{G})} \left(\frac{dz_l}{2\pi i z_l} \right) \Delta_{\mathbf{m}}(z_i)$$

$$\prod_{\alpha \in \mathrm{roots}} I_V \left(\mathbf{m}(\alpha), e^{ia(\alpha)} \right) \prod_{l=1}^{N_\chi} \prod_{w \in \rho_l} I_\chi^{(\rho_l)} \left(\mathbf{m}(w), e^{ia(w)} \right)$$

- The discrete 't Hooft fluxes are characterized by the following two quantities
$$UVU^{-1}V^{-1} = u, \quad V^r = v$$
with u, v lying in the center $Z(\hat{G})$, they project to flat configuration of $\hat{G}/Z(\hat{G})$.
- U, V are defined only up to multiplying center $\omega \in Z(\hat{G})$. Therefore
$$v \sim v\omega^r$$
- u also satisfies $u^r = 1$ because
$$(uV)^r = u^r V^r = (UVU^{-1})^r = UV^r U^{-1} = V^r \to u^r = 1.$$
- In particular, when the center is $\mathbb{Z}_N$, one has $u^N = v^N = 1$ such that
$$u^{\gcd(r,N)} = 1, \quad v \sim v\omega^{\gcd(r,N)}$$
- The index in the twist sector is similarly obtained by
$$\mathcal{I}[u, v] = \sum_{UVU^{-1}V^{-1}=u, V^r=v} \mathcal{I}_{U,V}$$

- There is only one closed 2-cycle $\Gamma_1 = C_\tau \times S^1$ and corresponding operators $U[\Gamma_1], \widetilde{U}[\Gamma_1]$. They satisfy

$$U[\Gamma_1]^r = U[\Gamma_1]^r = 1 \text{ and } U[\Gamma_1]^N = U[\Gamma_1]^N = 1$$

  and they combine to

$$U[\Gamma_1]^{\gcd(r,N)} = \widetilde{U}[\Gamma_1]^{\gcd(r,N)} = 1$$

- Those operators commute with each other since Γ_1 has no self-intersection number. It seems the Hilbert space is trivial

- Actually, one should include another 2-surface Γ_2 such that

$$\partial\Gamma_2 = rC_\tau$$

and consider the operators

$$U[\Gamma_2] = \exp\left[i \oint_{\Gamma_2} B\right], \quad \widetilde{U}[\Gamma_2] = \exp\left[i \oint_{\Gamma_2} \widetilde{B}\right]$$

- Since Γ_2 is not closed, one might worry they are not gauge invariant under the transformation

$$B \to B + d\lambda, \quad \widetilde{B} \to \widetilde{B} + d\widetilde{\lambda},$$

since by Stokes theorem

$$U[\Gamma_2] \to \omega^{ir\int_{C_\tau}\lambda} U[\Gamma_2], \quad \widetilde{U}[\Gamma_2] \to \omega^{ir\int_{C_\tau}\widetilde{\lambda}} \widetilde{U}[\Gamma_2]$$

- However, for level N BF theory both $B, \widetilde{B}$ and $\lambda, \widetilde{\lambda}$ are $\mathbb{Z}_N$-valued instead of $U(1)$-valued. One may check the following operators are gauge invariant

$$U[\Gamma_2]^{\frac{kN}{\gcd(r,N)}}, \widetilde{U}[\Gamma_2]^{\frac{\tilde{k}N}{\gcd(r,N)}}, \quad k, \tilde{k} = 0, \cdots, \gcd(r, N) - 1$$

- In summary, we have two kinds of operators generated by

$$\{U[\Gamma_1], U[\Gamma_2]^{\frac{N}{\gcd(r,N)}}\}, \quad \{\widetilde{U}[\Gamma_1], \widetilde{U}[\Gamma_2]^{\frac{N}{\gcd(r,N)}}\}$$

- The intersection number between Γ_1 and Γ_2 is one, therefore we have

$$\begin{cases} U[\Gamma_1]\widetilde{U}[\Gamma_2]^{\frac{N}{\gcd(r,N)}} = \omega^{-\frac{N}{\gcd(r,N)}}\widetilde{U}[\Gamma_2]^{\frac{N}{\gcd(r,N)}}U[\Gamma_1] \\ \widetilde{U}[\Gamma_1]U[\Gamma_2]^{\frac{N}{\gcd(r,N)}} = \omega^{+\frac{N}{\gcd(r,N)}}U[\Gamma_2]^{\frac{N}{\gcd(r,N)}}\widetilde{U}[\Gamma_1] \end{cases}$$

- The Dirichlet boundary state $|b_1, b_2\rangle$ is parameterized by two $\mathbb{Z}_N$-valued number b_1, b_2 satisfying

$$\gcd(r,N)b_1 = 0, \quad b_2 \sim b_2 + \gcd(r,N)$$

with

$$\begin{cases} U[\Gamma_1]|b_1, b_2\rangle = \omega^{b_1}|b_1, b_2\rangle \\ U[\Gamma_2]^{\frac{N}{\gcd(r,N)}}|b_1, b_2\rangle = \omega^{\frac{N}{\gcd(r,N)}b_2}|b_1, b_2\rangle \end{cases}$$

and

$$\begin{cases} \widetilde{U}[\Gamma_1]|b_1, b_2\rangle = |b_1, b_2 - 1\rangle \\ \widetilde{U}[\Gamma_2]^{\frac{N}{\gcd(r,N)}}|b_1, b_2\rangle = |b_1 - \frac{N}{\gcd(r,N)}, b_2\rangle \end{cases}$$

- The holonomies u, v are identified as

$$u = \omega^{b_1}, \quad v = \omega^{b_2}$$

and using $\gcd(r,N) b_1 = 0, b_2 \sim b_2 + \gcd(r,N)$ one recovers

$$u^{\gcd(r,N)} = 1, \quad v \sim v\omega^{\gcd(r,N)}$$

- The S/T-transformation acts separately as

$$\begin{cases} V_S|(b_1, b_2)\rangle = \frac{1}{\gcd(r,N)} \sum_{b_1', b_2' \in M_{r,N}} \omega^{b_1 b_2' + b_2 b_1'} |(b_1', b_2')\rangle \\ V_T|(b_1, b_2)\rangle = \omega^{\frac{1}{2} \int \mathfrak{P}(b)} |(b_1, b_2)\rangle \end{cases}$$

with

$$M_{r,N} = \left\{ b_1 = \frac{N k_1}{\gcd(r,N)}, b_2 = k_2 | k_1, k_2 \in \mathbb{Z}_{\gcd(r,N)} \right\}$$

- The dynamics boundary state is then constructed as

$$|\chi_{\text{SUSY}}\rangle = \sum_{b_1, b_2} \mathcal{I}[b_1, b_2] |b_1, b_2\rangle$$

Pontryagin square

- The Pontryagin square is

$$\begin{cases} \mathfrak{P}(b) = b \cup b + b \cup_1 \delta b \mod 2N & (N \text{ is even}) \\ \mathfrak{P}(b) = b \cup b \mod 2N & (N \text{ is odd}) \end{cases}$$

where

$$b = b_1\gamma_2 + b_2\gamma_1$$

and γ_1, γ_2 are Poincare dual of Γ_1, Γ_2 satisfying

$$\delta\gamma_1 = 0, \quad \delta\gamma_2 = r[C_\tau]$$

- The cup-1 product reads

$$\int \gamma_2 \cup_1 \delta\gamma_2 = r, \quad \int \gamma_1 \cup_1 \delta\gamma_2 = 0$$

which gives

$$\int \mathfrak{P}(b) = \begin{cases} 2b_1b_2 + rb_1^2 & (N \text{ is even}) \\ 2b_1b_2 & (N \text{ is odd}) \end{cases}$$

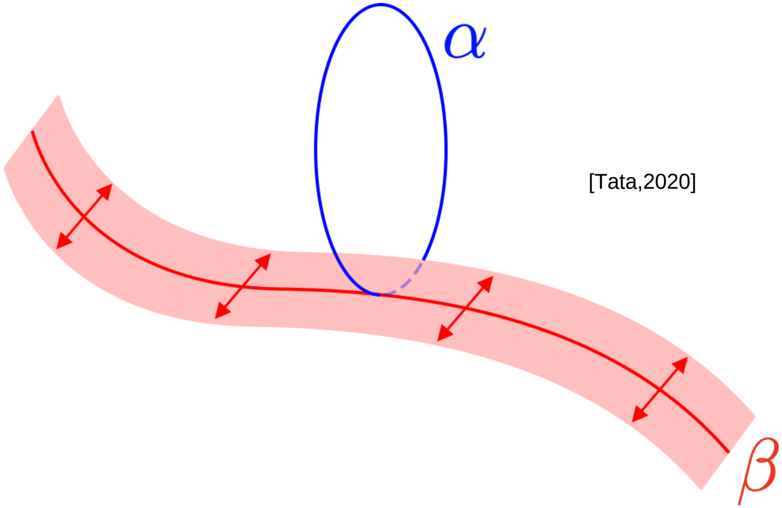


Figure: An illustration of the cup-1 product $\int [\alpha] \cup_1 [\beta]$ where $[\cdots]$ denote the Poincare dual. The thickening of β is given in both the positive and negative directions of the Morse flow (both directions pointing away from the central red curve). And $\int [\alpha] \cup_1 [\beta]$ measure the intersection between α and the thickening of β.

Conclusion

Changing global structure of gauge group

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Gauging 1-form symmetry/summing over 2-form background

$\updownarrow$

Switching topological boundary state in SymTFT

- We analyse the SymTFT formulated on various kinds of manifold, Spin, torsion, non-Spin.
- We use SymTFT to study the supersymmetric quantities of gauge theory, focusing on the global structure of gauge group.