On classification of theories with eight supercharges

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Motivation

Towards a classification of UV complete local field theories with 8 supercharges ($d \geq 3$), including 6d $(1,0)$ theories, 5d $\mathcal{N}=1$ theories, 4d $\mathcal{N}=2$ theories, and 3d $\mathcal{N}=4$ theories. (Two dimensional and one dimensional theories are special and we do not consider them here).

16 supercharges

The maximal number of SUSYs for local interacting field theory is 16. The classification results (modulo some global issues) are

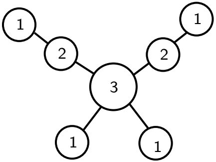
1. 6d $(2,0)$ theory is classified by ADE Lie algebra (Witten, 1995): two dimensional ADE singularities.
2. 4d $\mathcal{N}=4$ SYM is classified by simple Lie algebra.

These theories are well studied and major studying objects in new discoveries of field and string theory : AdS/CFT, integrability, bootstrap, amplitude, etc.

8 supercharges

Lots of examples have been collected for these theories in past 30 years.

- 3d UV complete theories are most easily found from gauge theory (Seiberg & Witten, 1996): gauge theories formed from free vector multiplets and free hypermultiplets are UV complete, and SCFTs are found as the IR fixed point of gauge theory. **Example**: quiver gauge theory.



- 4d theories are more constrained. There are gauge theories formed by vector multiplets and hypermultiplets: $SU(N)$ coupled with $N_f \leq 2N$. Many new SCFTs are found by using string theory construction:

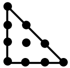
1. Quiver gauge theory

$$n_1 - SU(n) - SU(n) - n_2$$

2. Strongly coupled SCFT: Argyres-Douglas theory (Argyes & Douglas, 1995).
3. 6d $(2,0)$ construction: regular singularities (Gaiotto, 2009), and irregular singularities (D. Xie, 2012): Hitchin system.

4. IIB on 3-fold singularities, (Xie, Yau, 2015): $f(x,y,z,w) = 0$.

- 5d $\mathcal{N} = 1$ theories: gauge theory is not UV complete, and one define SCFTs by using string construction (Seiberg, 1996): M theory on a three-fold canonical singularity, 5-brane webs.



- 6d $(1, 0)$ theory is also only defined by string theory construction (Seiberg, 1996): F theory on an elliptic fibered local singularity. Example: $\frac{C^2 \times T^2}{\Gamma}$.

Perhaps the major lesson learned in recent years about SUSY field theory is that: One should not confine to the theory defined by conventional Lagrangian description (Wilson's point of view). String/M theory played a crucial role in most recent developments.

We did find a large class of new SUSY field theories with eight supercharges, mostly through string theory and geometric methods. The natural question is that: is it possible to have a classification like theories with 16 supercharges?

There are several comments about classification:

1. What do we mean by classification? In fact I do not aim a complete classification (which is always dangerous). What I want to do is to make the theory space as large as possible, and the methods of classification would help us to understand better these theories.
2. How can one proceed given that only the number of SUSYs are assumed? Given the large number of new theories, we already know that the constraint from SUSY itself is quite weak.

The extra constraints comes from deforming the theory and look at the theory in the IR limit (again Wilson's point of view of field theory). For SUSY field theory, we'd like to study the SUSY preserving deformations:

1. The relevant and marginal deformations are classified for theories with eight supercharges.
   1.1 6d theory: no SUSY deformation
   1.2 5d theory: mass deformation.
   1.3 4d theory: mass deformation, relevant and marginal.
   1.4 3d theory: mass deformation.
2. SUSY theory would have moduli space of vacua.

There are extra constraints coming from the structure of moduli space: one always gets more by considering a family of theories!

Dimensional reduction and unified treatment

The moduli space of 5d & 6d theory is parameterized by real scalars. To best use the power of complex and algebraic geometry, we compactify 5d & 6d theory on circle and torus so that one can get 4d $\mathcal{N} = 2$ theory in the low energy limit.

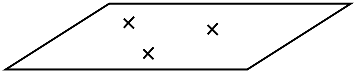
The low energy physics do receive contribution from higher dimensional theory and so it is possible to get the information of higher dimensional theory from 4d theory limit.

In our later studies, 3d $\mathcal{N} = 4$ theories would also play a crucial role in 4d Coulomb branch.

Coulomb branch

4d $\mathcal{N} = 2$ theories usually has a branch of moduli space, called Coulomb branch:

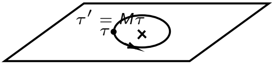
1. The low energy physics at generic point of Coulomb branch is abelian gauge theory: $U(1)^r$ gauge theory.
2. There are special points and the low energy physics is quite rich: it could be SCFT, IR free (abelian or non-abelian) gauge theory, or gauge theory coupled with strongly coupled matter.
3. A central goal is to determine the low energy physics at every vacua: determine the photon couplings at generic vacua, and the theory at special vacua.

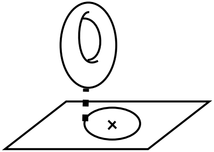


Seiberg-Witten solution

Seiberg-Witten (1994) found the solution on the Coulomb branch ($SU(2)$ gauge theory) by using following mathematical structure:

1. There is an extra torus attached to every vacua: the low energy coupling is given by complex structure of torus.
2. The physics of special vacua is given by the degeneration of the torus: the photon coupling is not single valued: electric-magnetic duality of abelian gauge theory. Geometrically, it is the monodromy group acting on homology.

1) 

2) 

The family of curves are expressed in a quite simple form:

$$y^2 = (x^n + \ldots + v_{n-1}x + v_n)^2 + \prod_{i=1}^{n_f}(x - m_i)$$

The SW curves for $SU(n)$ with $n_f \leq 2n$.

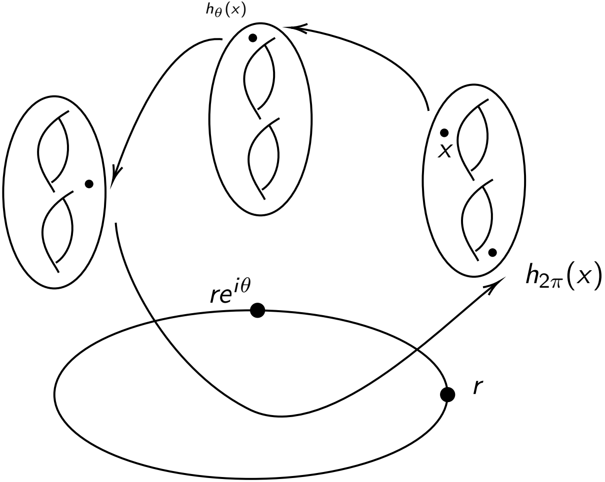
Seiberg-Witten found their solutions by using a series of deep physical insights (monopoles, instantons, anomalies, etc), which is difficult to be applied to other theories. The latter SW solutions are found through the connection to integral systems (Hitchin's system), and geometric method (singularity and deformation).

Strategy for classification

Our strategy of classification is to classify consistent SW geometry: namely family of algebraic varieties fibered over the parameter spaces. This is a hard problem if we follow the old ways of thinking about SW geometry. For example, it takes lots of efforts to classify 4d rank one SCFTs (Argyres et al, 2014). Until the September of 2022, I feel it is just hopeless to study the classification of general theories. There are three major new ideas that make it possible to go big steps forward:

The first idea is to focus on the classification of local singular type of one parameter families, and figure out the low energy IR theory. In the rank one case, Kodaira classified the singular fibers (8 types). In the rank two case, Namikawa-Ueno gave 140 types. In general, there is very nice theory by Matsumoto-Motositos. They showed that the local degeneration is given by the conjugacy class of the mapping class group of certain type.

Step 1: Local singularities



The classification of local singularity is a combinatorial game:

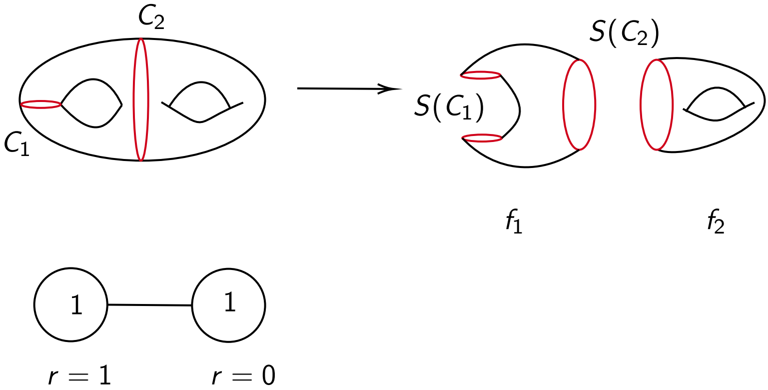


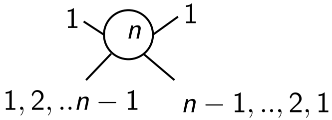
Figure: $S(C_i)$: negative screw numbers; f_i: periodic map.

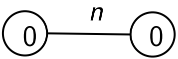
One can find low energy theory from the cutting system of the Riemann surface: a): Each cut gives a gauge group and fundamental hypers; b) Each component gives a matter system; c) This is quite similar to the usual Lagrangian description, and the theory description in class S theory.

Here it consists both SCFTs and IR free theories (notice that in Class S description the theory consists of UV complete theory and SCFTs).

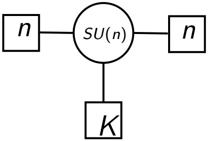
Some Examples:

$$(1)\ \left(n, g' = 0,\ \frac{1}{n} + \frac{1}{n} + \frac{n-1}{n} + \frac{n-1}{n}\right)$$



(2) 

$f = \frac{1}{n} + \frac{n-1}{n} + 1$ $\qquad$ $f = \frac{1}{n} + \frac{n-1}{n} + 1$



One can also read a configuration of irreducible curves from the combinatorial data

$$F = \sum_i n_i C_i$$

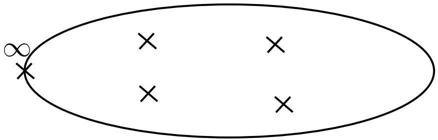
Link with 3d $\mathcal{N} = 4$ theories: I found that one can associate a 3d $\mathcal{N} = 4$ quiver gauge theory from the configuration of curves

1. There is a $U(n_i)$ gauge group for each C_i.
2. There is $n_{ij} = C_i \cdot C_j$ bi-fundamental hypermultiplets between $U(n_i)$ and $U(n_j)$ gauge groups.
3. There is g adjoint hyoermultiplets for $U(n_i)$.

These 3d theories quiver gauge theory gives rise to the 3d mirror for previous low energy theory.

Step 2: UV theory

So how to characterize the UV theory: whether it is a 5d theory, 6d theory, or 4d SCFT, asypmtotical free theory? The idea is to compactify the Coulomb branch, so that the moduli space becomes P^1, and an extra singularity is added to the ∞. The picture of the Coulomb branch geometry becomes



The property of the UV theory is reflected by the topological properties of the dual graphs of F_∞:

1. 4d theory: the dual graph is a tree of rational curves.
2. 5d theory: the dual graph is a chain of rational curves with one loop.
3. 6d theory: the dual graph is a chain of rational curves with two loops.

Interestingly, the number of loops is the same the number of loops on which the theory is compactified. We reached above result by considering the dimension of charge lattice: higher dimensional theory carries extra KK charges (besides the electric-magnetic charge).

We now has the following picture of SW geometry: there is a smooth surface $f : S \to P^1$, with singular fiber a collection of irreducible curves. The singular fiber at ∞ reflects the UV properties, and the configurations of curves gives the 3d mirror of the IR theory at special vacua, from which one can read the IR theory.

We call this SW picture the smooth model (relatively minimal). The usual ones given by singular curves are called singular model.

Step 3: Global SW geometry

Up to know, the consideration is mostly topological. The next step is the hardest one, namely describe the global SW geometry. The Coulomb branch geometry is constrained by special Kahler condition: equivalently a complete integrable system. For the hyperelliptic families, I found the solution to special Khaler condition:

1. First, we consider one parameter families $y^2 = f(x, t)$.
2. The general solution is given by replacing t by a polynomial encoding the Coulomb branch operators: $y^2 = f(x, P(u_i, x))$:

$$P(u_i, x) = \sum_i u_i x^i$$

3. The special Kahler condition puts constraints on the t dependence of $f(x, t)$.

Table: SW geometry for rank two 4d SCFTs

0		∞	
$y^2 = x^5 + t$	$(\frac{10}{7}, \frac{8}{7})$	$y^2 = t^2x + tx^6$	rank three $(10, 8, 6)$
$y^2 = x^5 + t^2$	$(\frac{5}{2}, \frac{3}{2})$	$y^2 = t^2x + x^6$	$(5, 3)$
$y^2 = x^6 + t$	$(\frac{6}{4}, \frac{5}{4})$	$y^2 = t^2 + tx^6$	rank three $(6, 5, 4)$
$y^2 = x^6 + t^2$	$(3, 2)$	$y^2 = x^6 + t^2$	$(3, 2)$
$y^2 = x^5 + xt^2$	$(4, 2)$	$y^2 = x^5 + xt^2$	$(4, 2)$
$y^2 = x^5 + xt$	$(\frac{8}{5}, \frac{6}{5})$	$y^2 = t^2x + x^5t$	$(8, 6)$
$y^2 = x^6 + xt$	$(\frac{5}{3}, \frac{4}{3})$	$y^2 = t^2 + tx^5$	$(5, 4)$
$y^2 = x^4t + t^2$	$(4, 3)$	$y^2 = t + x^4$	
$y^2 = x^4t + xt^2$	$(6, 4)$	$y^2 = t + x^3$	
$y^2 = tf_5(x)$	$(2, 2)$	$y^2 = txf_5^{'}(x)$	$(2, 2)$
$y^2 = x^5 + t^3$	$(10, 4)$	$y^2 = xt^4 + tx^6$	

Table: SW geometry for 4d SCFT

0		∞	
$y^2 = t^3 + m^6 x^5$	$(12, 6)$	$y^2 = t * (x^2 - t)^3 + t^4 x$	
$y^2 = t^3 + m^6 x^4$	$(6, 3)$	$y^2 = t * (x^2 - t)^3 + t^4 x^2$	
$y^2 = t^3 + m^4 t x^3$	$(8, 4)$	$y^2 = t * (x^2 - t)^3 + t^3 x (x^2 - t)$	
$y^2 = t^3 + m^6 x^3$	$(4, 2)$	$y^2 = t * (x^2 - t)^3 + t^4 x^3$	
$y^2 = t^3 + m^6 x^2$	$(3, 3/2)$	$y^2 = t * (x^2 - t)^3 + t^4 x^4$	
$y^2 = t^3 + m^4 t x$	$(\frac{8}{3}, \frac{4}{3})$	$y^2 = t * (x^2 - t)^3 + t^3 x (x^2 - t)$	
$y^2 = t^3 + m^6 x$	$(\frac{12}{5}, \frac{6}{5})$	$y^2 = t * (x^2 - t)^3 + t^4 x^5$	

Table: SW geometry for 6d $(1,0)$ theories

∞	*type*	0	
$xt + x^2(x-1)^2(x+1)$	I_{2-1-0}	$x^5t + t^2(x-1)^2(x+1)x$	
$xt + x^2(x-1)^2(x+1)^2$	I_{1-1-2}	$x^5t + t^2(x-1)^2(x+1)^2$	

The previous description of SW geometry is the conventional non-compact singular picture. How to translate the singular picture to the smooth compact picture, so that the IR and UV theory can be determined? The crucial ideas are following

1. The total geometry is compactified by treating $y^2 = f(x,t)$ as a double covering over the base $P^1 \times P^1$. One then add a singular fiber at $t = \infty$. Example: $f(x,t) = x^5 + t^2$, and the fiber at $t = \infty$ is $f(u,s) = su + u^6$, here the local coordinates around (∞, ∞) is (u,s).
2. There is a canonical resolution procedure (Horikawa, 1974) which will translate the local singularities into the smooth models.

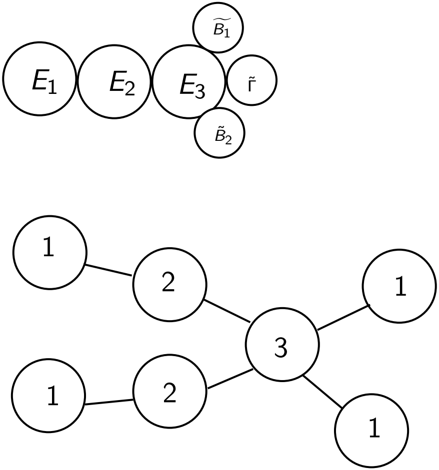


Figure: $f(x,t) = x^6 + t^2$

The formulation we found here are quite powerful:

1. It gives an efficient way of determining the IR theory for every vacua.
2. It gives a way of constructing new integrable systems (especially those corresponding to 5d and 6d theory). Connection to higher dimensional Painleve system.
3. Dynamical questions such as confinement ($\mathcal{N}=1$ deformation), chiral symmetry breaking, duality, etc.

Many further questions towards classification:

1. We assume the SW geometry is given by the family of Riemann surface. Namely, the abelian variety is given by the Jacobian of the Riemann surface. To find more solutions, one need to consider the families with symmetries to get more general abelian varieties (Prym variety).
2. It should be possible to consider SW geometry given by higher dimensional variety.
3. The SW geometry in the general case (not jus hyperelliptic families), and a systematical approach.
4. Theories formed by discrete gauging.

Thank you!