第五届全国场论与弦论学术研讨会

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Black holes from Supercharge Cohomology

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Talk based on collaboration with

- Li Feng, Ying-Hsuan Lin, Yi-Xiao Tao, Jingxiang Wu, Xi Yin 1305.6314, 2209.06728, 2306.04673, 2310.20086, 2402.10129

See also:

- Choi, Choi, Kim, Lee, Lee, Lee, Park [CCKLLLP] 2209.12696, 2304.10155, 2312.16443
- Budzik, Gaiotto, Kulp, Williams, Wu, Yu [BGKWWY] 2306.01039, and Budzik, Murali, Vieira [BMV] 2306.04693

Motivation and Backgrounds

- We would like to understand black holes and their microstates using the dual CFT in the **AdS/CFT correspondence**.

- AdS/CFT: A non-perturbative duality between quantum gravity in anti-de Sitter space (AdS) and conformal field theory (CFT) on the boundary.

- Main focus in this talk: **Large BPS (supersymmetric) black holes in** AdS_5 [Gutowski-Reall '04, Chong-Cvetic-Lu-Pope '05, …]. Their microstates are dual to BPS (supersymmetric) states in 4d maximal sper-Yang-Mills theory ($\mathcal{N} = 4$ SYM) with $\mathrm{SU}(N)$ gauge group.

- state/operator in CFT: BPS state $\overset{\text{1 to 1}}{\longleftrightarrow}$ BPS operators

Motivation

- The N^2 growth of the BPS black holes entropy $S = A/4G_N = \log(\#\,\mathrm{states})$ were reproduced by the superconformal index (a state counting with $(-1)^F$). [(Cabo-Bizet)-Cassani-Martelli-Murthy, Choi-Kim-Kim-Nahmgoong, Benini-Milan '18]

- Given a non-perturbatively complete framework of AdS/CFT, we should be able to answer more refined questions:
  - What are the wave functions and dynamics of the microstates?
  - How do we distinguish a weakly bound state of N^2 gravitons from typical black hole microstates?

- Supercharge cohomology reveals much richer information beyond SCI.

Motivation and Main Results

- New information beyond the superconformal index (SCI):
  - Complete BPS spectrum (It counts BPS states without $(-1)^F$.)
  - Information on the wave functions of BPS states (modulo exact terms in cohomology) and how they are related in theories at different ranks N.

- (One of the) main results: A classification of supercharge cohomology and their conjectural bulk duals [CC-Lin '24]:
  - Monotone (graviton) cohomology $\longleftrightarrow$ smooth horizonless geometry
  - Fortuitous (BH) cohomology $\longleftrightarrow$ typical black hole microstate

Outline

- Introduction to supercharge cohomology
- A classification: monotone (graviton) and fortuitous (black holes)
- Bulk duals of supercharge cohomologies
- Preliminary results on D1D5 CFT

Supercharge Cohomology

Supercharge Cohomology

- Supercharge cohomology can be defined very generally. It only needs a supercharge Q that is nilpotent $Q^2 = 0$.

  The main focus in this talk:

  4d maximal SYM with SU(N) gauge group

- Pick a pair Q & $S = Q^\dagger$ out of 16 Q and 16 S.

  BPS bound: $\Delta = 2\{Q, Q^\dagger\} = E - (J_1 + J_2 + q_1 + q_2 + q_3) \geq 0$

  BPS states saturate the bound. (spins J_i , R-charges q_i)

Supercharge Cohomology

- Nilpotency $Q^2 = 0$.

$$Q\text{-cohomology} = \frac{\{O \,|\, QO = 0\}}{\{O \,|\, O = QO'\}}$$

- Hodge theory argument:

$$Q\text{-cohomology classes} \overset{\text{1 to 1}}{\longleftrightarrow} \text{BPS states } (\Delta = 2\{Q, Q^\dagger\} = 0)$$

(Recall: de Rham cohomology classes $\overset{\text{1 to 1}}{\longleftrightarrow}$ harmonic forms)

Supercharge Cohomology

- The non-renormalization conjecture: Q-cohomology is independent of the Yang-Mills coupling g_{YM} as long as $g_{\text{YM}} \neq 0$.
  - Evidence:
    1. Matching BPS spectrum at infinite N (see later)
    2. Consistency with S-duality of N=4 SYM [**CC**-Choi-Dong-Yan WIP]
- We will compute Q-cohomology at weak coupling $g_{\text{YM}} \ll 1$.

Weak-Coupling Setup

- At weak couplings, local operators could be written in terms of multitraces of fundamental fields with covariant derivatives (both $N \times N$ matrices) and modulo trace relations.

- Trace relations play an important role in the study of supercharge cohomology. A simple example of trace relations is

  e.g. for any 2×2 matrix X, $2\mathrm{Tr}\, X^3 = 3\mathrm{Tr}\, X\, \mathrm{Tr}\, X^2 - (\mathrm{Tr}\, X)^3$.

- A property of trace relations we will use later:
  - I_N = (space of trace relations for $N \times N$ matrices) . We have $I_{N+1} \subsetneq I_N$

Weak-Coupling Setup

- At weak coupling, it suffices to work with fundamental fields that are BPS in free theory. They can be assembled into a superfield $\Psi(z^\alpha, \theta_i)$ on superspace $\mathbb{C}^{2|3}$ with two bosonic coordinates z^α ($\alpha = \pm$) and three fermionic coordinates θ_i ($i = 1{,}2{,}3$). [CC-Yin '13]

$$\Psi(z^\alpha, \theta_i) \sim \lambda_\alpha z^\alpha + \phi^i \theta_i + \epsilon^{ijk} \psi_i \theta_j \theta_k + F_{++} \theta^3 + \cdots,$$

λ_α gauginos
$\phi^i = (X, Y, Z)$ complex scalars
ψ_i complex fermions
F_{++} self-dual field strength
$(\cdots)$ covariant derivatives

- The Q action takes a very concise form

$$Q(\Psi) = \Psi^2 \quad \text{and} \quad Q(AB) = Q(A)B \pm AQ(B)$$

A Classification of Cohomologies: Monotone and Fortuitous

A Classification of Cohomologies

- Some definitions/notations:
  - $\widetilde{\mathscr{H}}$: the space of formal multi-traces (without imposing trace relations)
  - $\mathscr{H}_N$: space of local operators in the $\mathrm{SU}(N)$ theory
  - I_N : space of trace relations at rank N

- A short exact sequence (SES): ($\mathscr{H}_N \simeq \widetilde{\mathscr{H}} / I_N$)

$$0 \to I_N \xrightarrow{i} \widetilde{\mathscr{H}} \xrightarrow{\pi} \mathscr{H}_N \to 0$$

i : inclusion map, π : quotient map that imposes the trace relations

A Classification of Cohomologies

- Taking Q-cohomology, the SES induces a long exact sequence (LES):

$$\cdots \to H^n_Q(I_N) \xrightarrow{i} H^n_Q(\widetilde{\mathscr{H}}) \xrightarrow{\pi} H^n_Q(\mathscr{H}_N) \xrightarrow{\delta} H^{n+1}_Q(I_N) \to \cdots$$

$$\mathrm{im}\,(i) = \ker\,(\pi)\,, \quad \mathrm{im}\,(\pi) = \ker\,(\delta)$$

n: a cohomological grading (the number of the superfield Ψ)

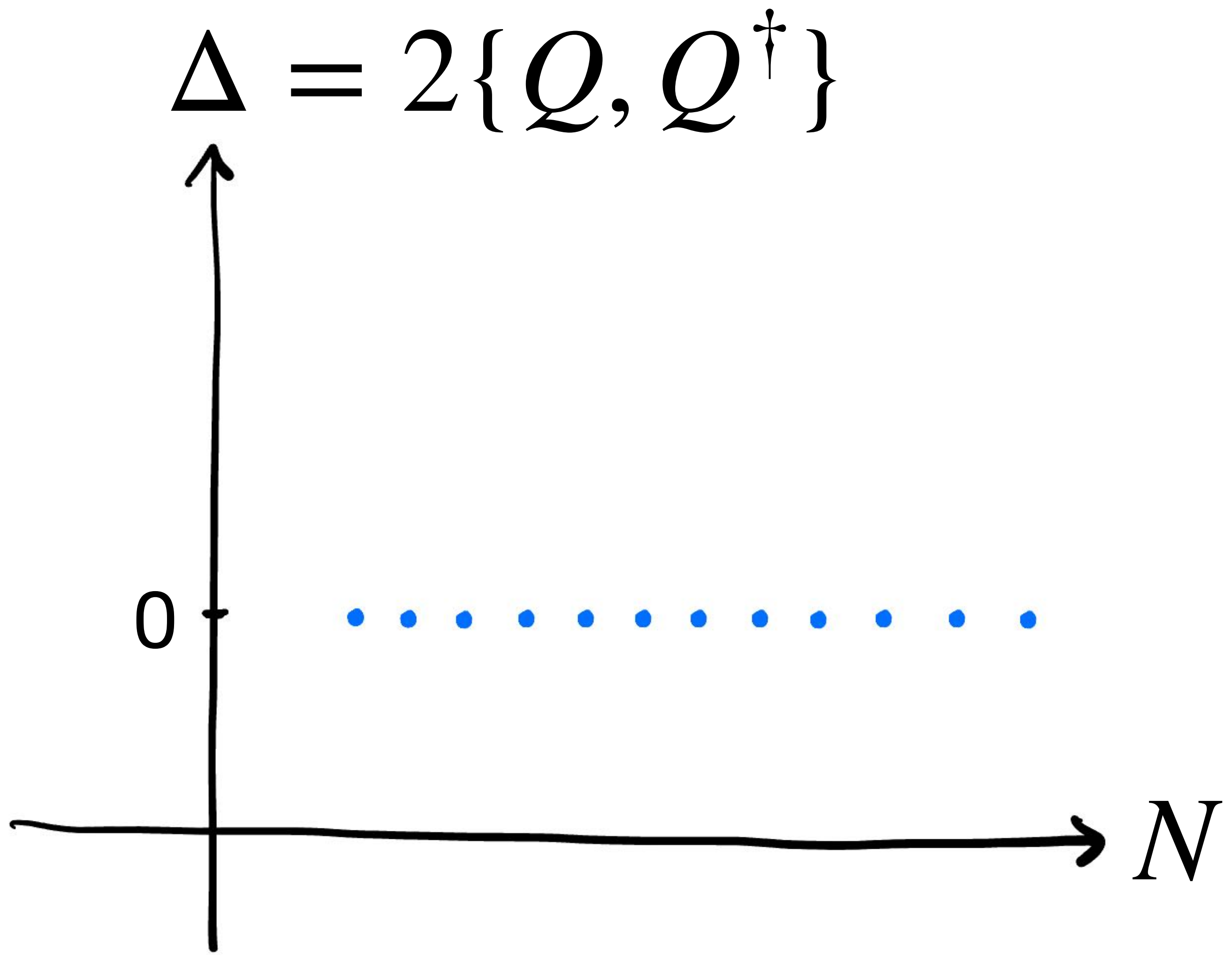
- A classification of cohomologies [CC-Lin '24]:
  - Monotone (graviton) cohomology = $\mathrm{im}\,\pi \simeq H^n_Q(\widetilde{\mathscr{H}})/\mathrm{im}\,i$
  - Fortuitous (black hole) cohomology = $\mathrm{im}\,\delta \simeq H^n_Q(\mathscr{H}_N)/\mathrm{im}\,\pi$

Large N sequences of operators

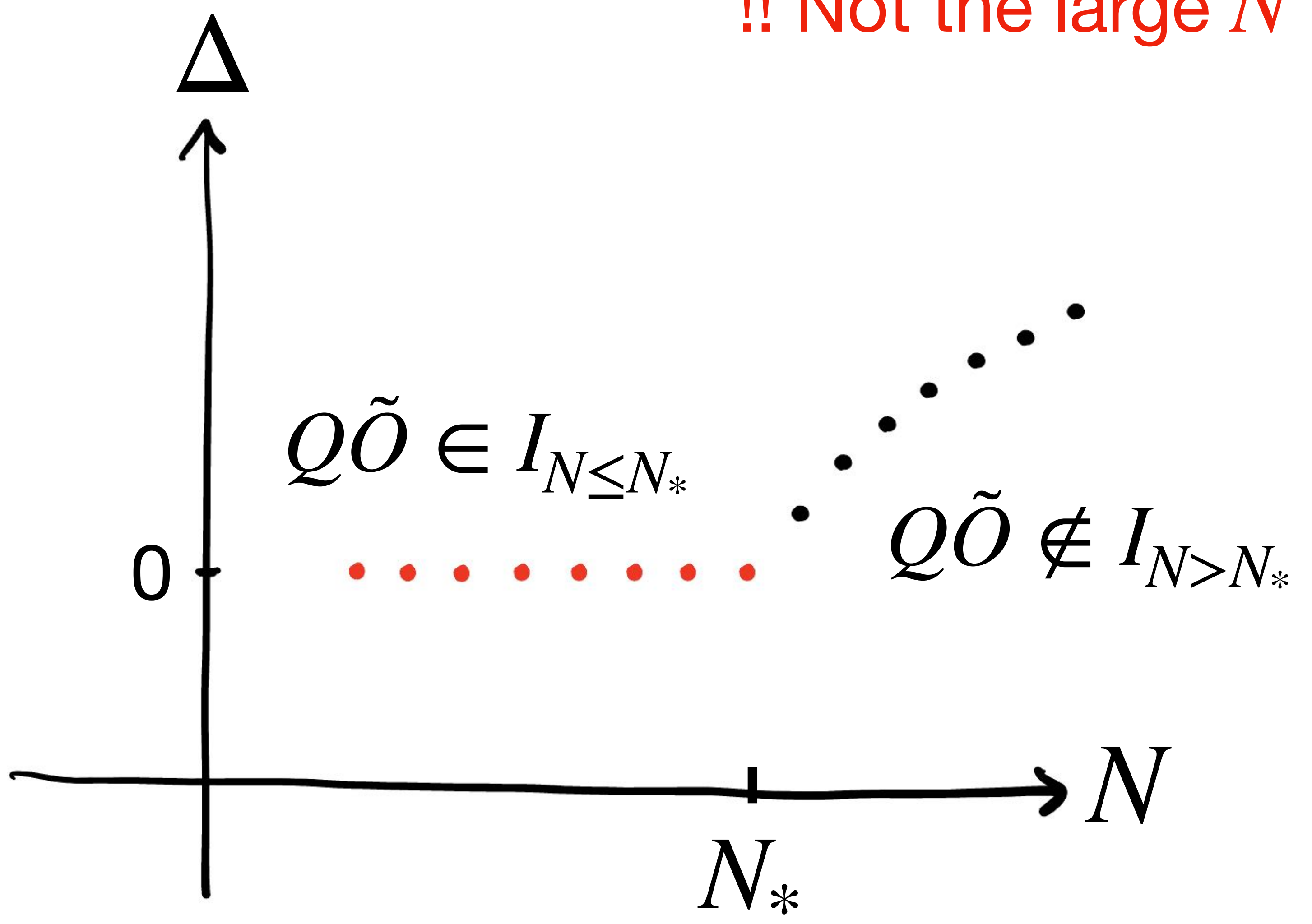
- Let O represent a Q-cohomology class, and write O non-uniquely as a multitrace $\tilde{O}$.

- Monotone (graviton) cohomology:
  - $Q\tilde{O} = 0$ w/o using trace relations
  - Admit infinite N limits with **fixed** $\tilde{O}$

- Fortuitous (black hole) cohomology:
  - $Q\tilde{O} =$ (a nontrivial trace relation)
  - No infinite N limit (with fixed $\tilde{O}$)



!! Not the large N limit with $E \sim N^2$!!



Monotones in $\mathcal{N} = 4$ SYM

- Consider one-forms on the superspace $\mathbb{C}^{2|3}$: basis dz^{α}, $d\theta_i$

$$d\Psi \equiv dz^{\dot{\alpha}}\,\partial_{z^{\dot{\alpha}}}\Psi + d\theta_i\,\partial_{\theta_i}\Psi$$

Supercharge action: $Qd\Psi = [\Psi, d\Psi]$

- The multitrace $\mathrm{Tr}\,[(d\Psi)^{n_1}]\cdots\mathrm{Tr}\,[(d\Psi)^{n_L}]$ is Q-closed and not Q-exact without using trace relations.

- All monotone Q-cohomologies could be obtained by imposing trace relations at finite N on $\mathrm{Tr}\,[(d\Psi)^{n_1}]\cdots\mathrm{Tr}\,[(d\Psi)^{n_L}]$. [CC-Yin '13] [BGKWWY '23]

- All BPS op's (1/16 BPS) ⊃ monotone op's ⊃ 1/8 BPS op's [CC-Lin-Wu '23]

Fortuitous in $\mathcal{N} = 4$ SYM

- By a brute force comprehensive search in the SU(2) theory up to high spin and R-charges, we found the first fortuitous Q-cohomology. [CC-Lin '22]
  - Very hard to find (1 in 10^5 cohomology classes) (Doesn't mean fortuitous are few!)
  - Explicit representative [Choi-Kim-Lee-Park '22] : $\partial^{i_1 \cdots i_n} \equiv \partial_{\theta_{i_1}} \cdots \partial_{\theta_{i_n}}$

$$O = \epsilon_{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3} \epsilon_{l_1 l_2 l_3} \epsilon_{m_1 m_2 m_3} \epsilon^{k_1 l_1 m_1} \mathrm{Tr}(\partial^{i_1} \Psi \partial^{k_2 k_3} \Psi) \mathrm{Tr}(\partial^{j_1} \Psi \partial^{l_2 l_3} \Psi) \mathrm{Tr}(\partial^{i_2 i_3} \Psi \partial^{j_2 j_3} \Psi \partial^{m_2 m_3} \Psi)$$

- Working in the BMN sector (only ∂_{θ_i} and no $\partial_{z^{\dot{\alpha}}}$), [CCKLLLP '22, '23] performed a more efficient search and achieved the following results:
  - SU(2) and SU(3): multiple infinite towers of fortuitous cohomologies
  - SU(4): leading fortuitous cohomology

Bulk Duals of Q-cohomologies

Bulk Duals of Monotones

- While supercharge cohomology is independent of the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$, we will focus on large λ and look for bulk duals in supergravity.

- Conjecture: Monotone Q-cohomologies at infinite N are dual to BPS multi-particles in $\mathrm{AdS}_5 \times S^5$.
  - Evidence: The counting of the multitraces $\mathrm{Tr}\,[(d\Psi)^{n_1}]\cdots\mathrm{Tr}\,[(d\Psi)^{n_L}]$ matches with the BPS multi-graviton partition function. [CC-Yin '13]

Bulk Duals of Monotones

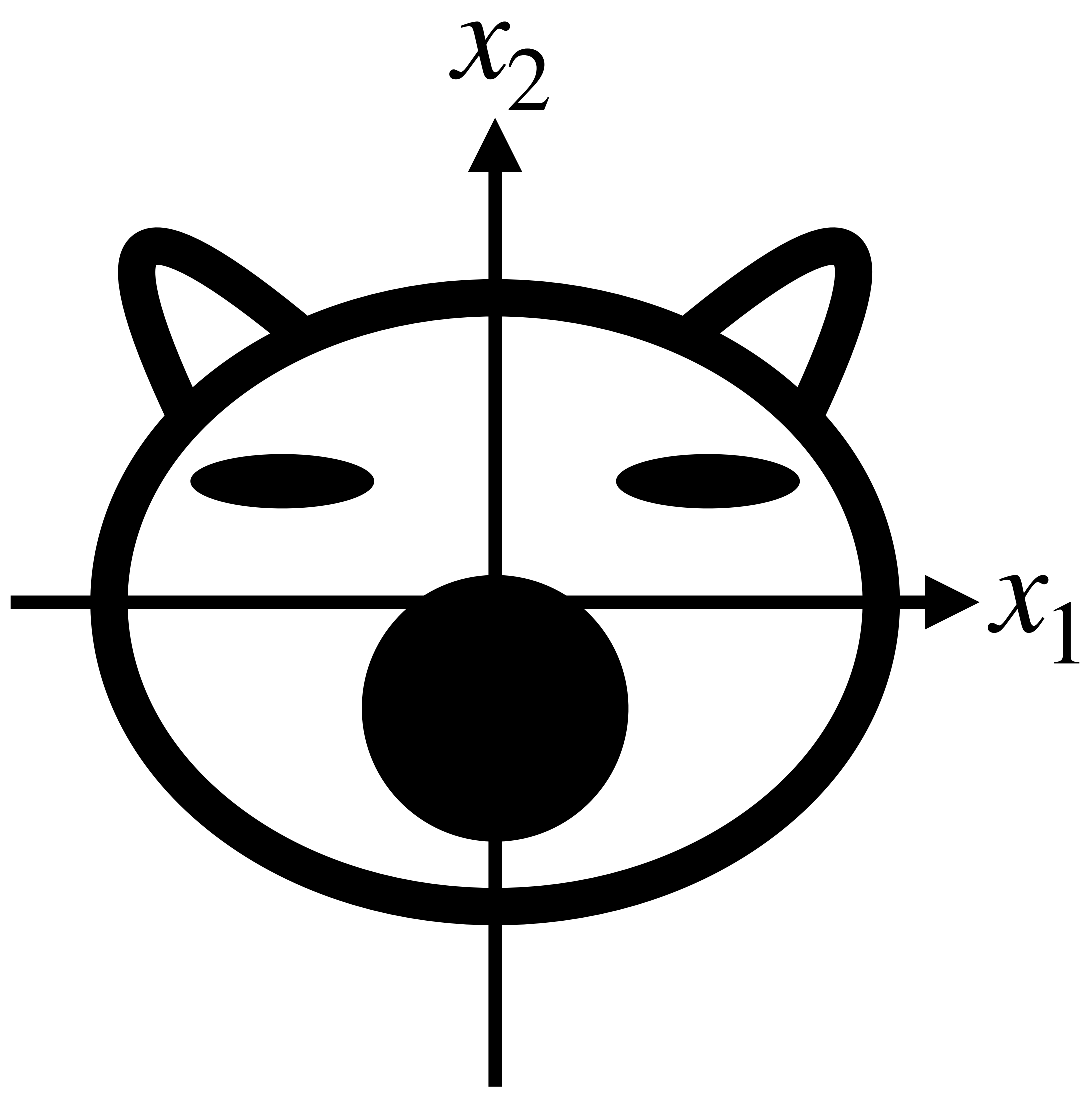
- Conjecture [CC-Lin '24] : Monotone Q-cohomologies at finite N are dual to quantizations of smooth horizonless solutions in supergravity.
  - An intuitive argument: Consider $G_N \to 0$ $(N \to \infty)$ limit with fixed spins, charges, and energy of a smooth horizonless solution. It is expected to remain smooth horizonless and become perturbative particles in AdS.
  - Evidence: BPS operators in the $\mathrm{SU}(2\,|\,3)$ sector (1/8 BPS) are dual to (generalizations) of Lin-Lunin-Maldacena (LLM) geometries. [LLM '04, ...] Furthermore, focusing on 1/2-BPS operators, the partition function can be reproduced by quantizing the Lin-Lunin-Maldacena (LLM) geometries. [Grant-Maoz-Marsano-Papadodimas-Rychkov '05, CC-Lin '24]

Half-BPS operators

- Half-BPS operators $\mathrm{Tr}\,X^{m_1}\cdots\mathrm{Tr}\,X^{m_L}$, complex scalar $X=\phi^1$
  - trace relations: $\mathrm{Tr}\,X^{N+m}=$ (multitraces) one for each $m>0$
  - $QX=0 \;\Rightarrow$ all half-BPS operators are monotone
  - Partition function: $Z_{\text{half-BPS}}=\underbrace{\left(\prod_{m=1}^{\infty}\frac{1}{1-q^m}\right)}_{\mathrm{Tr}\,X^m}\times\underbrace{\left(\prod_{m=1}^{\infty}(1-q^{N+m})\right)}_{\text{trace relations}}$
- Classically, half-BPS operators are dual to the LLM geometries, which are smooth horizonless half-BPS geometries.

LLM Geometries

- The space of LLM geometries is $\mathscr{M} = \{u(x) \in \mathbb{Z}_2 \,|\, x \in \mathbb{R}^2\}$. [LLM '04]



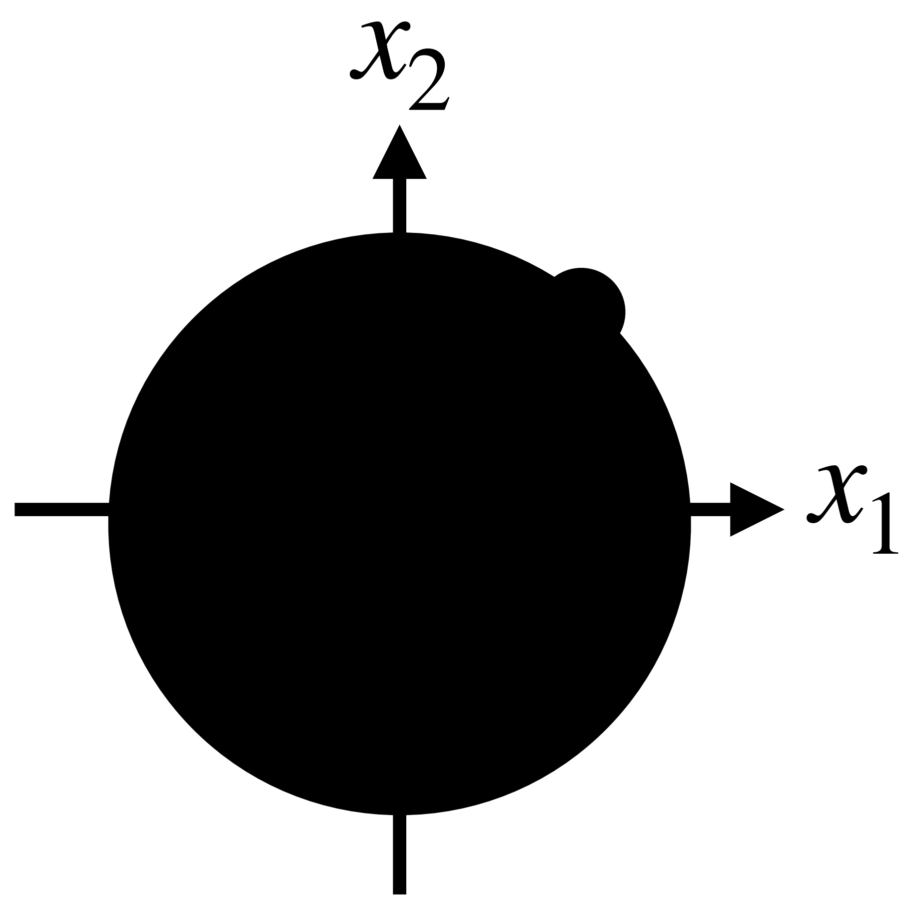
Total black area: $A = 2\int_{S^5} F_5 = 2\pi\kappa\sqrt{\hbar_{10}}N$

ADM mass: $H = \dfrac{1}{4\pi\kappa^2}\displaystyle\int (x_1^2 + x_2^2)u(x)d^2x - \dfrac{1}{8\pi^2\kappa^2}A^2$

- $\mathscr{M}$ is equipped with a symplectic form determined by IIB supergravity.

Quantization of LLM Geometries

- LLM Geometries can be quantized by “promoting the symplectic form to a commutator”.
- [GMMPR ’05] computed the symplectic form with the small fluctuation approximation, and the quantization recovered the half-BPS partition function in the infinite N limit.
- We are not satisfied with this.



Quantization of LLM Geometries

- Let us quantize the LLM geometries without any approximation. [CC-Lin ’24]

- The exact Poisson bracket can be obtained by a consistency condition [Rychkov ‘05]: The Hamiltonian dynamics on the classical moduli space must agree with the symmetries of the system. In our case, we have

$$\frac{du(x,t)}{dt} = \{H, u(x,t)\} \quad \Leftrightarrow \quad \frac{du(x,t)}{dt} = \frac{du(x,t)}{d\phi}$$

from BPS condition $H = q_1$

- Since we know H, we could use the consistency condition to find $\{\,\cdot\,,\,\cdot\,\}$.

Quantization of LLM Geometries

- Exact Poisson bracket: $A[u] = \int a(x)u(x)d^2x$, similar for $B[u]$

$$\{A[u], B[u]\} = 2\pi\kappa^2 \int \left(\frac{\partial a}{\partial x_1}\frac{\partial b}{\partial x_2} - \frac{\partial b}{\partial x_1}\frac{\partial a}{\partial x_2} \right) u(x)d^2x$$

- “promoting the symplectic form to a commutator” means finding a quantum system whose commutator reduces to the Poisson bracket as $\hbar_{10} \to 0$.

- Free fermions: $\{\psi^\dagger(x), \psi(y)\} = \hbar^{-1}\delta(x-y),$ $u(x) = \hbar e^{\frac{i}{\hbar}xp}\psi^\dagger(x)\psi(p)$

$\hbar = \kappa\sqrt{\hbar_{10}}$

Quantization of LLM Geometries

$$\psi(x) = \hbar^{-\frac{1}{2}} \sum_{n=0}^{\infty} c_n \Psi_n(x) \, , \quad A = 2\pi\hbar \sum_{n=0}^{\infty} c_n^{\dagger} c_n = 2\pi\hbar N, \quad H = \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) c_n^{\dagger} c_n - \frac{N^2}{2}$$

Partition function: $Z = \operatorname{Tr} e^{-\beta H} = \prod_{n=1}^{N} \frac{1}{1 - e^{-\beta\hbar_{10} n}}$

- Reproduce the finite N half-BPS partition function!

- Generalization: quantization of other BPS geometries [WIP].

Bulk Duals of Fortuitous?

- There are exponentially more fortuitous Q-cohomologies than monotone ones.

A typical black hole microstate is fortuitous.

- A bound on the number of monotone Q-cohomologies:

$$\boxed{\begin{array}{l}\text{\# monotones}\\ \text{at finite } N\end{array}} \underset{\substack{\uparrow\\ \text{trace relations}}}{<} \boxed{\begin{array}{l}\text{\# monotones}\\ \text{at infinite } N\end{array}} = \boxed{\begin{array}{l}\text{\# BPS multi}\\ \text{particle states}\end{array}} < \boxed{\begin{array}{l}\text{\# all multi}\\ \text{particle states}\end{array}} \sim e^{E^{\frac{9}{10}}} \sim \underset{\substack{\uparrow\\ \text{BH energy } E \sim N^2}}{e^{N^{\frac{9}{5}}}}$$

$\nearrow$ Entropy of gas of free particles $\sim E^{\frac{9}{10}}$

- The growth of the total number of all BPS states can be estimated by the superconformal index [CCMM, CKKN, BM ’18].

$$\text{(\# monotones) + (\# fortuitous) = (\# all BPS)} \sim e^{N^2}$$

What is a typical black hole microstate?

- A black hole is a mixed state. A fortuitous BPS state is dual to a single black hole microstate, which is a typical state among the pure states that constitute the mixed state.

- From the property of fortuitous BPS states, we know that a typical BPS black hole microstate should become non-BPS as $G_N \to 0$. It is hard to imagine any smooth horizonless BPS geometry that could exhibit this property. Hence, the bulk dual of fortuitous is likely to involve strings and branes on top of supergravity backgrounds.

- This proposal is different from the standard Strominger-Vafa paradigm that describes black hole microstates as D-branes at $\lambda \ll 1$.

Preliminary results on D1D5 CFT

D1D5 CFT

- D1D5 CFTs have a conformal manifold with an orbifold point as $\mathrm{Sym}^N \mathscr{M}$ with $\mathscr{M} = T^4,\, K3$.

- Twisted sectors of $\mathrm{Sym}^N \mathscr{M}$ are labeled by the S_N conjugacy classes $(1)^{N_1}(2)^{N_2}\cdots(m)^{N_m}$. Each twisted sector (n) is given by the S_N invariant Fock space generated by oscillators (magnons) of $\mathscr{M}$.

$$(n) \quad \leftrightarrow \quad \mathrm{Tr}\,(\cdots)$$

oscillators ↔ letters

Q-action in D1D5 CFT

- The supercharge cohomology of the D1D5 CFT is conjectured to be invariant along the conformal manifold away from the orbifold points.

- The Q-action is complicated.
  - Need to be computed at least at the leading order in the conformal perturbation theory.
  - It does not satisfy the Leibniz rule, and may mix different cycles.
  - It was computed recently in the large N in [Gaberdiel-Gopakumar-Nairz '23].

“Trace Relations” in D1D5 CFT

- The role of the trace relations is played by the stringy exclusion principle, which simply constrains the total length of the nontrivial twisted cycles to be less than N.

- Trivial cycle: (1) without any excitations (similar to $\mathrm{Tr}\,(1)$).
  Example: $(1)^{N_1}(2)^{N_2}\cdots(m)^{N_m}$, with $N_1' \leq N_1$ copies of nontrivial (1). The stringy exclusion principle is $N_1' + 2N_2 + \cdots + mN_m \leq N$.

- The Q-action commutes with the stringy exclusion principle. Hence, monotone and fortuitous cohomologies are well-defined.

D1D5 CFT

- Again, we expect the monotone cohomologies to be dual to smooth horizonless BPS geometries.
- The largest class of such geometries are called superstrata [Bena-Giusto-Russo-Shigemori-Warner '15, …]
- A point of similarity to N=4 SYM: [Shigemori '19]

$$(\#\ \text{superstrata}) < (\#\ \text{multi-particle states}) \ll (\#\ \text{BH microstates})$$

Future Direction

- Construct fortuitous Q-cohomology at larger N.
  - So far we only have examples of $N = 2, 3, 4$
  - It is maybe enough to go up to $N \gtrsim 6$, because S/N^2 computed from the superconformal index already shows convergence at around $N \sim 6$.

- Generalizations: supercharge cohomology in D1-D5 CFTs [**CC**-Lin-Zhang WIP], 4d N=2 SCFTs [**CC**-Choi-Dong-Yan WIP], BMN matrix quantum mechanics, …

- BPS states in BMN ↔ BPS black holes (strings) in M-theory pp wave bg.
  - Witten index exhibits black hole entropy growth

Thank you